Algebra for generalised quantum observables

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Abstract

Generalised observables (POM observables) are necessary for representing *all* possible measurements on a quantum system. Useful algebraic operations such as addition and multiplication are defined for these observables, recovering many advantages of the more restrictive Hermitian operator formalism. Examples include new uncertainty relations and metrics, and optical phase applications.

1 Introduction

The assertion that *all* observable quantities of a quantum system can be represented by Hermitian operators acting on the Hilbert space of the system is now well known to be inconsistent with the usual representation of measurement via interaction between the system and an apparatus. In particular, consider the measurement procedure defined by preparing an apparatus in some fixed initial state, allowing an interaction between the system and apparatus, and measuring some Hermitian operator on the apparatus. The probability of a given measurement result, a, for initial state $|\psi\rangle$ of the system, is then found to have the general form

$$p(a|\psi) = \langle\psi|A_a|\psi\rangle, \tag{1}$$

where A_a is a positive operator acting on the system Hilbert space, determined by the specific measurement setup (i.e., by the initial apparatus state, the interaction Hamiltonian, and the apparatus observable).

The set of operators $\{A_a\}$ corresponding to such a measurement procedure is called a probability operator measure (POM) or a positive operator valued measure (POVM), and the corresponding physical observable of the system, $\mathcal{A}$, may be referred to as a POM observable [1, 2, 3, 4]. In general A_a is *not* a projection operator associated with an eigenspace of some corresponding Hermitian operator.

Thus, to consider *all* possible measurements on a quantum system (without reference to specific experimental setups), one must consider all POM observables on the Hilbert space of the system. Such observables represent a non-trivial extension of Hermitian observables, being necessary to describe, for example, optimal measurements for distinguishing between non-orthogonal states [1, 2, 4, 5], and for describing optical phase [1, 2, 4, 6, 7, 8, 9] and optical heterodyne detection [4, 10, 11].

However, what is gained in generality is lost in algebraic simplicity. For example, it is not *a priori* clear how to add and subtract, much less multiply, POM observables. It would therefore be desirable to be able to define an algebra on the class of POM observables, to regain some of the advantages of the operator algebra associated with (the less general) Hermitian observables. The aim of this Letter is to demonstrate the existence of such an algebra, and applications thereof.

In the following section POM observables are briefly reviewed, and maximal and non-redundant observables defined. In section 3 the sum, product and other binary operations are defined for the relatively simple case of one POM observable and one Hermitian observable, and examples of statistical deviation and distance given in section 4. In section 5 the algebraic combination of two arbitrary POM observables is considered. This general case is more difficult, as certain consistency conditions must be observed, but is solvable. A generalised uncertainty relation and optical phase examples are given in section 6, and conclusions in section 7.

2 Generalised observables

The requirement that the probability distribution $p(a|\psi)$ in Eq. (1) is positive and normalised for all states ψ implies that the POM $\{A_a\}$ satisfies

$$A_a \geq 0, \qquad \sum_a A_a = 1. \tag{2}$$

In fact, any set of operators $\{A_a\}$ satisfying these conditions may be realised by a measurement procedure of the type discussed in the Introduction, and hence Eq. (2) fully characterises the class of POM observables [1, 2, 3, 4]. The summation is replaced by integration over continuous ranges of measurement outcomes.

From Eq. (1) the expectation value of any function $f(\mathcal{A})$ of a general POM observable $\mathcal{A}$ is given by

$$\langle\, f(\mathcal{A})\,\rangle = \sum_a f(a)\, p(a|\psi) = \langle\psi|\overline{f(\mathcal{A})}|\psi\rangle, \tag{3}$$

where $\overline{f(\mathcal{A})}$ is defined to be the operator

$$\overline{f(\mathcal{A})} := \sum_a f(a) A_a. \tag{4}$$

Hermitian observables correspond to the special case $A_a A_{a'} = 0$ for $a \neq a'$ (and real-valued outcomes). For such observables the associated operator $A = \overline{\mathcal{A}}$ is Hermitian, and satisfies $f(A) = \overline{f(\mathcal{A})}$ for any function f. Further, one has $A_a - A_a^2 = A_a \sum_{a'\neq a} A_{a'} = 0$, i.e., for Hermitian POMs each A_a is a projection operator (onto an eigenspace associated with eigenvalue a of A).

Now, for any positive operator A_a appearing in some POM, a new POM can be obtained by replacing A_a by N copies of A_a/N (i.e., by $A_{a,1}, \ldots, A_{a,N}$, with $A_{a,i} \equiv A_a/N$). However, this new POM observable is trivially measured, by making a measurement of the original observable and throwing an N-sided die if outcome a is obtained (to distinguish the N new possibilities). Thus this new POM is *physically redundant*, only differing from the original in a trivial statistical sense that is independent of the actual system. A similar redundancy is obtained if the weightings $1/N$ are replaced by any set of positive numbers $w_1, \ldots, w_N$ which sum to unity.

A POM observable $\mathcal{A}$ is therefore called *non-redundant* if $A_a \neq \lambda A_b$ for $a \neq b$ and any real number λ. Note that any POM observable may be trivially (and uniquely) reduced to a non-redundant POM observable, by adding together any proportional terms in the corresponding POM. Note also that non-redundant observables exclude the trivial possibility $A_a = 0$.

Finally, a POM observable $\mathcal{A}$ is called *maximal* if $A_a = |a\rangle\langle a|$ for all a (the kets $\{|a\rangle\}$ are not necessarily orthogonal nor normalised). It may be shown that maximal POMs are maximally informative, in the sense that any measurement which optimises information gain under a given constraint is equivalent to the measurement of some maximal POM. Further, since any positive operator A_a may be decomposed into a sum of "back-to-back" kets, it follows that every POM observable has a (non-unique) maximal extension. Attention will therefore be restricted to non-redundant maximal POM observables in what follows.

3 Combinations of POM and Hermitian observables

Defining the algebraic combination of a maximal POM observable with any Hermitian observable is relatively straightforward, and hence worth examining separately from the general case. The key here is the promotion of the POM observable to an equivalent Hermitian operator on an extended Hilbert space, in such a way that all algebraic relations between existing Hermitian operators are preserved.

In particular, let H denote the Hilbert space of the system, and for a maximal POM observable $\mathcal{A} \equiv \{|a\rangle\langle a|\}$ let $H_{\mathcal{A}}$ denote the Hilbert space of square-integrable functions over the space of outcomes of $\mathcal{A}$. There is then an orthonormal basis $\{|a)\}$ for H_A associated with these outcomes, and a natural mapping from H to H_A defined by

$$|\psi\rangle \to |\psi_{\mathcal{A}}) := \sum_a \langle a|\psi\rangle \, |a). \tag{5}$$

From Eqs. (1) and (5) one has

$$p(a|\psi) = |\langle a|\psi\rangle|^2 = |(a|\psi_{\mathcal{A}})|^2.$$

Thus the POM $\{|a\rangle\langle a|\}$ on H has statistics equivalent to the Hermitian operator

$$\hat{A} := \sum_a a \, |a)(a| \tag{6}$$

on $H_{\mathcal{A}}$, where the latter corresponds to the *Hermitian* POM $\{|a)(a|\}$.

Note that the representation $\psi(a) = \langle a|\psi\rangle = (a|\psi_{\mathcal{A}})$ is well known for particular POM observables (eg, the coherent state and canonical phase representations [2, 8]). The associated representation of $\mathcal{A}$ as a Hermitian operator $\hat{A}$ on H_A provides a simple example of Naimark's extension theorem [1, 2, 4] for the (admittedly trivial) case of maximal POMs.

The set of extended states, $\{|\psi_{\mathcal{A}})\}$, is characterised by the projection operator

$$E := \sum_{a,a'} \langle a|a'\rangle |a)(a'| = E^2. \tag{7}$$

In particular, it may be checked that any normalised ket $|\psi)$ in $H_{\mathcal{A}}$ is generated by some physical state $|\psi\rangle$, as per Eq. (5), if and only if

$$E|\psi) = |\psi).$$

Thus the unit eigenspace of E is isomorphic to the physical Hilbert space H. Note that for a *Hermitian* observable one has $E \equiv 1$, and hence in this case (and only in this case) $H_{\mathcal{A}}$ is isomorphic to H.

Further, let X be any Hermitian operator on H. Then there is a natural extension of X to a Hermitian operator $X_{\mathcal{A}}$ on $H_{\mathcal{A}}$, defined by

$$X_{\mathcal{A}} = \sum_{a,a'} \langle a|X|a'\rangle \, |a)(a'|. \tag{8}$$

It may be checked that (i) the product XY is mapped to $X_{\mathcal{A}}Y_{\mathcal{A}}$, and (ii) the state $X|\psi\rangle$ is mapped to $X_{\mathcal{A}}|\psi_{\mathcal{A}})$. Thus this extension preserves all algebraic properties of Hermitian operators and states on H.

Finally, there is an inverse mapping of states and observables from $H_{\mathcal{A}}$ to H, generated by the projection $|a) \to |a\rangle$. It follows from Eqs. (5), (6) and (8) that $|\psi_{\mathcal{A}}) \to |\psi\rangle$ and $X_{\mathcal{A}} \to X$ under this mapping, and that the Hermitian POM $\{|a)(a|\}$ corresponding to $\hat{A}$ is mapped to the original POM $\{|a\rangle\langle a|\}$. Moreover, an arbitrary POM $\{|k)(k|\}$ on $H_{\mathcal{A}}$ is mapped to the POM observable $\{|k\rangle\langle k|\}$ on H, where

$$|k\rangle := \sum_a (a|k)|a\rangle. \tag{9}$$

Note that orthonormality of the basis $\{|a)\}$ implies

$$\begin{aligned} \sum_k |k\rangle\langle k| &= \sum_{k,a,a'} (a|k)(k|a')|a\rangle\langle a'| \\ &= \sum_{a,a'} (a|a')|a\rangle\langle a'| \\ &= \sum_a |a\rangle\langle a| = 1, \end{aligned}$$

as required by Eq. (2).

The tools for defining algebraic combinations of $\mathcal{A}$ and X (such as the sum $\mathcal{A}+X$, the symmetric product $(\mathcal{A}X + X\mathcal{A})/2$, and the "commutator" $i[\mathcal{A}, X]$), are now laid out. In particular, let g denote *any* function which maps pairs of Hermitian operators to a Hermitian operator. Hence,

$$\hat{K} := g(\hat{A}, X_{\mathcal{A}}) \tag{10}$$

is a Hermitian operator on $H_{\mathcal{A}}$, with an associated Hermitian POM $\{|k)(k|\}$. The corresponding mapping $g(\mathcal{A}, X)$, for a general POM observable $\mathcal{A}$ and Hermitian observable X on H, is then defined to be the POM observable

$$g(\mathcal{A}, X) := K \equiv \{|k\rangle\langle k|\}, \tag{11}$$

where $|k\rangle$ is defined by Eq. (9).

Note from Eqs. (5) and (9) that $\langle k|\psi\rangle = (k|\psi_{\mathcal{A}})$. Hence the probability distributions $p(k|\psi)$ and $p(k|\psi_{\mathcal{A}})$ are equivalent, and from Eqs. (10) and (11) one therefore has the identity

$$\langle\, g(\mathcal{A}, X)\,\rangle = (\psi_{\mathcal{A}}|g(\hat{A}, X_{\mathcal{A}})|\psi_{\mathcal{A}}) = \sum_{a,a'} \langle\psi|a\rangle\,(a|g(\hat{A}, X_{\mathcal{A}})|a')\,\langle a'|\psi\rangle \qquad (12)$$

for expectation values. This is a very useful formula, as it means one can calculate the expectation value of $g(\mathcal{A}, X)$ without having to explicitly determine the corresponding POM $\{k\rangle\langle k|\}$ (which would require diagonalising the Hermitian operator $\hat{K} = g(\hat{A}, X_{\mathcal{A}})$).

4 Examples: deviation and distance

From Eq. (12) one may calculate the statistical deviation between $\mathcal{A}$ and X, for a given state $|\psi\rangle$, as

$$\begin{aligned}
\langle\,(X-\mathcal{A})^2\,\rangle &= (\psi_{\mathcal{A}}|X_{\mathcal{A}}^2 + \hat{A}^2 - X_{\mathcal{A}}\hat{A} - \hat{A}X_{\mathcal{A}}|\psi_{\mathcal{A}}) \\
&= \langle X^2\rangle + \langle \mathcal{A}^2\rangle - \sum_{a,a'}\Big[(\psi_{\mathcal{A}}|a)(a|X_{\mathcal{A}}|a')(a'|\hat{A}|\psi_{\mathcal{A}}) + c.c.\Big] \\
&= \langle\psi|(X-\overline{\mathcal{A}})^2|\psi\rangle + \langle\psi|\overline{\mathcal{A}^2} - (\overline{\mathcal{A}})^2|\psi\rangle, \qquad (13)
\end{aligned}$$

where Eqs. (5), (6) and (8) have been used, and the operators $\overline{\mathcal{A}}$, $\overline{\mathcal{A}^2}$ are defined via Eq. (4). This quantity provides a measure of the degree to which the POM $\mathcal{A}$ may be approximated by a given Hermitian operator X. Note that the second term is *not* equal to the variance of $\mathcal{A}$. One also has the alternative formula

$$\begin{aligned}
\langle\,(X-\mathcal{A})^2\,\rangle &= \sum_a (\psi_{\mathcal{A}}|(X_{\mathcal{A}} - \hat{A})|a)(a|X_{\mathcal{A}} - \hat{A}|\psi_{\mathcal{A}}) \\
&= \sum_a |\langle a|X - a|\psi\rangle|^2,
\end{aligned}$$

which has been used previously (without algebraic justification) in the context of obtaining exact uncertainty relations between photon number and optical phase [12].

Further, the "distance" between a POM observable $\mathcal{A}$ and a Hermitian observable X on H may be defined via a natural generalisation of the Hilbert-Schmidt metric:

$$\begin{aligned}
d(\mathcal{A}, X)^2 &:= \mathrm{tr}[\,(\hat{A} - X_{\mathcal{A}})^2\,] \\
&= \mathrm{tr}[(X-\overline{\mathcal{A}})^2 + \overline{\mathcal{A}^2} - (\overline{\mathcal{A}})^2]. \qquad (14)
\end{aligned}$$

This generalised metric satisfies the triangle inequalities

$$d(\mathcal{A},X)+d(\mathcal{A},Y)\geq d(X,Y)\geq|d(\mathcal{A},X)-d(\mathcal{A},Y)|$$

as an automatic consequence of the Hilbert-Schmidt metric on $H_{\mathcal{A}}$, and hence $d(\mathcal{A},X)$ may indeed be interpreted as a measure of "distance".

From Eq. (14) one has the Pythagorean relation

$$d(\mathcal{A},X)^2=d(\mathcal{A},\overline{\mathcal{A}})^2+d(\overline{\mathcal{A}},X)^2$$

between $\mathcal{A}$, X and $\overline{\mathcal{A}}$. It follows immediately that $\overline{\mathcal{A}}$ represents the Hermitian operator "closest" to $\mathcal{A}$, being separated from $\mathcal{A}$ by the minimum distance

$$\min_X d(\mathcal{A},X)=d(\mathcal{A},\overline{\mathcal{A}})=\{\mathrm{tr}[\overline{\mathcal{A}^2}-(\overline{\mathcal{A}})^2]\}^{1/2}.$$

This minimum distance is a useful measure of the inherent "fuzziness" of $\mathcal{A}$, vanishing if and only if $\mathcal{A}$ is a Hermitian observable.

5 Combining arbitrary POM observables

To generalise the above results to algebraic combinations of two maximal POMs $\mathcal{A}\equiv\{|a\rangle\langle a|\}$, $\mathcal{B}\equiv\{|b\rangle\langle b|\}$, one must find a Hilbert space $H_{\mathcal{AB}}$ which contains two suitable orthonormal sets $\{|a)\}$, $\{|b)\}$. It turns out that, to avoid such undesirable properties such as $\mathcal{A}-\mathcal{A}\neq 0$, the compatibility of $\mathcal{A}$ and $\mathcal{B}$ must explicitly be taken into account. The resulting simultaneous mapping of $\mathcal{A}$ and $\mathcal{B}$ to Hermitian operators on $H_{\mathcal{AB}}$ corresponds to a novel and highly nontrivial Naimark extension, in contrast to the mapping from H to $H_{\mathcal{A}}$ defined in section 3.

First, note that each state $|\psi\rangle$ on H will have two possible extensions on $H_{\mathcal{AB}}$, given by $|\psi_{\mathcal{A}})$ and $|\psi_{\mathcal{B}})$ as calculated via Eq. (5), corresponding to respective projection operators $E_{\mathcal{A}}$ and $E_{\mathcal{B}}$ defined as per Eq. (7). Consistency requires that these are equal. Calculating $(b|\psi_{\mathcal{A}})$ and $(a|\psi_{\mathcal{B}})$ via Eq. (5) then yields the conditions

$$|a\rangle=\sum_b(b|a)|b\rangle,\quad |b\rangle=\sum_a(a|b)|a\rangle \tag{15}$$

on the inner product $(a|b)$.

Second, since the statistics of $\mathcal{A}$ and $\mathcal{B}$ on H will be equivalent to those of the Hermitian operators $\hat{A}$ and $\hat{B}$ on $H_{\mathcal{AB}}$ respectively, as calculated via

Eq. (6), it is physically desirable that any *identical* statistical components of $\mathcal{A}$ and $\mathcal{A}$ are mapped to identical components of $\hat{A}$ and $\hat{B}$, i.e.,

$$|a)(a| = |b)(b| \quad \text{for} \quad |a\rangle\langle a| = |b\rangle\langle b|. \tag{16}$$

This condition ensures that compatible aspects of the observables $\mathcal{A}$ and $\mathcal{B}$ are preserved by the extension to $H_{\mathcal{AB}}$ (and in particular that $\hat{A} = \hat{B}$ for $\mathcal{A} = \mathcal{B}$).

If conditions (15) and (16) can be satisfied, then any function g which maps pairs of Hermitian operators to a Hermitian operator will define a Hermitian operator

$$\hat{K} := g(\hat{A}, \hat{B}),$$

on $H_{\mathcal{AB}}$, with an associated Hermitian POM $\{|k)(k|\}$. The corresponding POM observable $g(\mathcal{A}, \mathcal{B})$ on H is then defined to be the POM $K \equiv \{|k\rangle\langle k|\}$, with

$$|k\rangle := \sum_a (a|k)|a\rangle = \sum_b (b|k)|b\rangle$$

in analogy to Eq. (9), where the second equality follows immediately from Eq. (15).

It therefore remains to find orthonormal sets $\{|a)\}$, $\{|b)\}$ satisfying Eqs. (15) and (16). First, let $C := \{|a\rangle\langle a|\} \cap \{|b\rangle\langle b|\} \equiv \{|c\rangle\langle c|\}$ denote the set of *common* elements of the POMs corresponding to $\mathcal{A}$ and $\mathcal{B}$. C is physically well-defined for non-redundant POMs, where measurement of the POM $C \cup \{C_0\}$, with

$$C_0 := 1 - \sum_c |c\rangle\langle c|,$$

corresponds to the simultaneous measurement of the compatible components of $\mathcal{A}$ and $\mathcal{B}$. Condition (16) implies that these compatible components are promoted to simultaneous eigenstates of $\hat{A}$ and $\hat{B}$.

Decomposing the POMs for $\mathcal{A}$ and $\mathcal{B}$ as $\{|c\rangle\langle c|, |\tilde{a}\rangle\langle\tilde{a}|\}$ and $\{|c\rangle\langle c|, |\tilde{b}\rangle\langle\tilde{b}|\}$ respectively, it follows that

$$\sum_{\tilde{a}} |\tilde{a}\rangle\langle\tilde{a}| = \sum_{\tilde{b}} |\tilde{b}\rangle\langle\tilde{b}| = C_0. \tag{17}$$

Eqs. (15) and (16) then imply that the orthonormal set $\{|c)\}$ is orthogonal to each of the orthonormal sets $\{|\tilde{a})\}$, $\{|\tilde{b})\}$ on $H_{\mathcal{AB}}$, and further that the inner product $(\tilde{a}|\tilde{b})$ satisfies

$$|\tilde{a}\rangle = \sum_{\tilde{b}} (\tilde{b}|\tilde{a})|\tilde{b}\rangle, \quad |\tilde{b}\rangle = \sum_{\tilde{a}} (\tilde{a}|\tilde{b})|\tilde{a}\rangle.$$

These equations are formally solved by the choice

$$(\tilde{a}|\tilde{b}) = \langle\tilde{a}|C_0^{-1}|\tilde{b}\rangle \tag{18}$$

as may easily be checked using Eq. (17). However, it must be verified that this solution has all the properties required of an inner product between two orthonormal (not necessarily complete) sets in some Hilbert space $\tilde{H}_{\mathcal{AB}}$ (one may then take $H_{\mathcal{AB}} = \tilde{H}_{\mathcal{AB}} \oplus H_C$, where H_C is the span of $\{|c)\}$).

First, each of $|\tilde{a}\rangle$ and $|\tilde{b}\rangle$ are orthogonal to the zero-eigenspace of C_0 (since Eq. (17) implies that $\langle\tilde{a}|C_0|\tilde{a}\rangle$ and $\langle\tilde{b}|C_0|\tilde{b}\rangle$ are strictly positive), and hence the righthand side of Eq. (18) is well-defined. Second, noting C_0 is Hermitian, then $(\tilde{a}|\tilde{b}) = (\tilde{b}|\tilde{a})^*$ as required. Finally, for it to be possible to write each $|\tilde{a})$ as an orthogonal superposition of the $|\tilde{b})$ and some orthonormal set $\{|x)\}$, i.e.,

$$|\tilde{a}) = \sum_{\tilde{b}} (\tilde{b}|\tilde{a})|\tilde{b}) + \sum_x (x|\tilde{a})|x),$$

one must have $\sum_{\tilde{b}} |(\tilde{b}|\tilde{a})|^2 \leq 1$ (and similarly $\sum_{\tilde{a}} |(\tilde{b}|\tilde{a})|^2 \leq 1$). But

$$\sum_{\tilde{b}} |(\tilde{b}|\tilde{a})|^2 = \langle\tilde{a}|C_0^{-1}|\tilde{a}\rangle = \langle\overline{a}|\overline{a}\rangle$$

from Eq. (18), where $|\overline{a}\rangle := C_0^{-1/2}|\tilde{a}\rangle$. Noting from Eq. (17) that $\sum_{\overline{a}} |\overline{a}\rangle\langle\overline{a}|$ is equivalent to the unit operator on the span of $\{|\overline{a}\rangle\}$, one then has

$$\begin{aligned} \langle\overline{a}|\overline{a}\rangle &= \langle\overline{a}|[\sum_{\overline{a}'} |\overline{a}'\rangle\langle\overline{a}'|]|\overline{a}\rangle \\ &= \sum_{\overline{a}'} |\langle\overline{a}|\overline{a}'\rangle|^2 \geq \langle\overline{a}|\overline{a}\rangle^2, \end{aligned}$$

implying $\langle\overline{a}|\overline{a}\rangle \leq 1$ as required.

6 Examples: uncertainty relations and phase

The expectation value (and hence the statistics) of any algebraic combination of two arbitrary POMs $\mathcal{A}$ and $\mathcal{B}$ on H may now be calculated via the relation

$$\langle\, g(\mathcal{A}, \mathcal{B})\,\rangle = (\psi|g(\hat{A}, \hat{B})|\psi) \tag{19}$$

analogous to Eq. (12), where $|\psi)$ denotes either of $|\psi_{\mathcal{A}})$, $|\psi_{\mathcal{B}})$.

The examples of statistical deviation and distance in section 4 generalise immediately to such pairs of POM observables. As a further example, the expectation of the "commutator" of $\mathcal{A}$ and $\mathcal{B}$ follows as

$$\begin{aligned}\langle\, i[\mathcal{A},\mathcal{B}]\,\rangle &= i(\psi|[\hat{A},\hat{B}]|\psi) \\ &= i(\psi|[\sum_{a,b} ab(a|b)|a)(b| - h.c.]|\psi) \\ &= i\langle\psi|[\sum_{\tilde{a},\tilde{b}} \tilde{a}\tilde{b}\langle\tilde{a}|C_0^{-1}|\tilde{b}\rangle|\tilde{a}\rangle\langle\tilde{b}| - h.c.]|\psi\rangle \\ &= i\langle\psi|\tilde{A}C_0^{-1}\tilde{B} - \tilde{B}C_0^{-1}\tilde{A}|\psi\rangle,\end{aligned}$$

where the operators

$$\tilde{A} := \sum_{\tilde{a}} \tilde{a}|\tilde{a}\rangle\langle\tilde{a}|, \quad \tilde{B} := \sum_{\tilde{b}} \tilde{b}|\tilde{b}\rangle\langle\tilde{b}|$$

represent the restrictions of $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ to non-identical $|a\rangle\langle a|$ and $|b\rangle\langle b|$. The Heisenberg uncertainty relation on $H_{\mathcal{AB}}$ therefore leads immediately to the *generalised* uncertainty relation

$$\Delta\mathcal{A}\,\Delta\mathcal{B} \geq \frac{1}{2}|\langle\psi|\tilde{A}C_0^{-1}\tilde{B} - \tilde{B}C_0^{-1}\tilde{A}|\psi\rangle| \tag{20}$$

for arbitrary POM observables $\mathcal{A}$ and $\mathcal{B}$ on H.

In the case that $\mathcal{A}$ and $\mathcal{B}$ have *no* POM components in common, then $C_0 = 1$ and the above uncertainty relation reduces to

$$\Delta\mathcal{A}\,\Delta\mathcal{B} \geq \frac{1}{2}|\langle\psi|[\overline{\mathcal{A}},\overline{\mathcal{B}}|\psi\rangle|.$$

As an example, for the photon number operator $N = \sum_n n|n\rangle\langle n|$ and canonical phase POM $\Phi \equiv \{|\phi\rangle\langle\phi|\}$, where $|\phi\rangle = (2\pi)^{-1/2}\sum_n e^{in\phi}|n\rangle$ and $\phi \in (-\pi,\pi]$ [1, 2, 4, 6, 7, 8, 9], a straightforward calculation yields

$$\begin{aligned}\Delta N\Delta\Phi &\geq (1/2)|\langle\psi|[N,\overline{\Phi}]|\psi\rangle| \\ &= (1/2)|\langle\psi|\sum_{m,n}(-1)^{m+n}|m\rangle\langle n| - 1|\psi\rangle| \\ &= (1/2)|1 - 2\pi p(\pi|\psi)|,\end{aligned}$$

in agreement with previous (non-algebraic) methods [8, 12, 13].

Another phase observable of interest arises from ideal heterodyne detection, where one makes a simultaneous (but noisy) measurement of the quadratures of a single-mode optical field [4, 10, 11]. This measurement is represented by the the so-called "coherent-state" POM $\mathcal{A}_H \equiv \{\pi^{-1}|\alpha\rangle\langle\alpha|\}$, where $|\alpha\rangle$ denotes the coherent state corresponding to eigenvalue $\alpha = re^{i\phi}$ of the photon annihilation operator a. Thus $\Phi_H := \arg \mathcal{A}_H$ is a corresponding phase observable: the *heterodyne* phase [4, 14, 15] (see also Ref. [16] for a different realisation of this observable).

Since phase is a periodic variable, the *circular* deviation

$$\delta_H := 1 - |\langle\, e^{i\Phi} e^{-i\Phi_H}\,\rangle|$$

provides a more natural measure of comparison for Φ and Φ_H than does $\langle(\Phi - \Phi_H)^2\rangle$ (even better is $1 - |\langle e^{i(\Phi-\Phi_H)}\rangle|$, but this is more difficult to evaluate). This quantity will be close to zero in cases where heterodyne phase provides a good approximation to the canonical phase, and close to unity in cases where heterodyne phase is a poor estimate of the canonical phase. Since δ_H is defined by an algebraic function of two POM observables, it can be calculated by the above methods.

In particular, from Eq. (19) one has $\langle\, \mathcal{A}\mathcal{B}\,\rangle = \langle\psi|\overline{\mathcal{A}}\;\overline{\mathcal{B}}|\psi\rangle$ when $\mathcal{A}$ and $\mathcal{A}$ share no common components. A straightforward calculation gives

$$\overline{e^{i\Phi}} = \sum_n |n\rangle\langle n+1|,$$

while a Gaussian integration yields [15, 16])

$$\begin{aligned}\overline{e^{-i\Phi_H}} &= \pi^{-1}\int d^2\alpha\, e^{-i\phi}|\alpha\rangle\langle\alpha| \\ &= \sum_n \Gamma(n+3/2)(n!)^{-1}(n+1)^{-1/2}|n+1\rangle\langle n|,\end{aligned}$$

and hence one finds

$$\delta_H = 1 - \sum_n |\langle n|\psi\rangle|^2 \Gamma(n+3/2)(n!)^{-1}(n+1)^{-1/2}. \tag{21}$$

The circular deviation between the canonical and heterodyne phases is therefore completely determined by the photon number distribution of the state. Further, Stirling's formula for the Gamma function may be used to obtain the asymptotic formula

$$\delta_H \sim \langle\psi|(N+1)^{-1}|\psi\rangle/8$$

from Eq. (21), to first order in $1/(N+1)$. Hence δ_H is typically small for high energy states, implying that heterodyne detection provides an accurate estimate of the canonical phase for such states.

7 Conclusions

It has been shown how to define algebraic operations for arbitrary pairs of generalised quantum observables. As well as being of intrinsic interest, this allows many of the advantages of calculations with Hermitian operators to be realised for the more general POM observables, as indicated by the examples in sections 4 and 6.

The algebraic combination of any two POM observables $\mathcal{A}$ and $\mathcal{B}$ with any number of Hermitian observables $X, Y, Z, \ldots$ may also be defined, with Eqs. (12) and (19) generalising to

$$\langle\, g(\mathcal{A}, \mathcal{B}, X, Y, Z, \ldots)\,\rangle = (\psi_{\mathcal{AB}}|g(\hat{A}, \hat{B}, X_{\mathcal{AB}}, Y_{\mathcal{AB}}, Z_{\mathcal{AB}}, \ldots)|\psi_{\mathcal{AB}}),$$

where $|\psi_{\mathcal{AB}})$ denotes either of $|\psi_{\mathcal{A}}) \equiv |\psi_{\mathcal{B}})$ and $X_{\mathcal{AB}}$ denotes either of $X_{\mathcal{A}} \equiv X_{\mathcal{B}}$ (the second equivalence follows from the first, via $E_{\mathcal{A}} \equiv E_{\mathcal{B}}$). Note that for the special case where $\mathcal{A}$ and $\mathcal{B}$ correspond to two Hermitian operators A and B respectively, then $H_{\mathcal{AB}}$ is isomorphic to H, and so $g(\mathcal{A}, \mathcal{B}, X, Y, Z, \ldots)$ becomes equivalent to the Hermitian operator $g(A, B, X, Y, Z, \ldots)$.

A number of issues remain for future investigation. First, only binary combinations of POM observables have been considered. It is not clear whether general combinations of three or more such observables can be consistently defined, nor even whether, for example, the operation of addition is associative. Second, the question of uniqueness has not been examined. It is possible there are other solutions satisfying conditions (15) and (16), and even that one could satisfactorily replace the second of these conditions with a weaker (and smoother) requirement along the lines that $\hat{A}$ is "close" to $\hat{B}$ whenever $\mathcal{A}$ is "close" to $\mathcal{B}$. Third, it might be possible to use the existence of an algebra for POM observables to provide a "cleaner" formulation of QM, not requiring any *a priori* distinction between POM observables and Hermitian observables (which, for example, the usual mapping between classical and quantum Hamiltonians requires). For example, the optical phase observable can now simply be defined *algebraically*, in direct analogy to the classical formula, as the combination $e^{i\Phi} := N^{-1/2}a$.

Applications of the generalised statistical deviation and distance, discussed in section 4, to determining the optimal estimate of an observable from a given measurement and to finding general "disturbance" and "joint-measurement" uncertainty relations, have recently been given (since this paper was first prepared) [17, 18].

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